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THE POSSIBILITY OF AVOIDING AN ENCOUNTER IN A LINEAR DIFFERENTIAL GAME OF EVASION†

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A linear differential game of evasion is considered. The possibility of avoiding an encounter is proved when the condition of "rotatability" and the condition for the inclusion of the domain of values of the controls of the pursuer into the domain of values of the controls of the evader are satisfied. © 2002 Elsevier Science Ltd. All rights reserved.

The formulation of the problem being considered is due to Pontryagin [1, 2] and many investigations have been concerned with it (see [3–10]). Pontryagin's theorem requires that two conditions are satisfied: the conditions of "rotatability" and "advantage". A number $\mu > 1$, which reflects the advantage of the evading object over the pursuing object, participates in the formulation of the second condition. In 1973, Pontryagin formulated the problem: is it possible to prove the theorem of evasion [2] if $\mu = 1$? However, a positive solution of the problem has still not been obtained.

A solution of Pontryagin's problem is given below in the case when the domain of control of the pursuer is contained in the domain of control of the evader.

1. STATEMENT OF THE PROBLEM AND FORMULATION OF THE RESULTS

A control vector z is considered, the motion of which is described by the linear differential equation

$$\dot{z} = Cz - u + v + a;\ z \in R^n,\ u \in P,\ v \in Q \tag{1.1}$$

C is a linear mapping of R^n onto itself, a is a specified constant vector from R^n, u is the control parameter of the pursuer, v is the control parameter of the evader, and P and Q are specified convex, non-empty compact subsets of the space R^n. The game is considered to be completed if z reaches a specified linear subspace M of the space R^n. The aim of the evader is to prevent the termination of the game, and to achieve that a value $v(t)$ of the parameter v can be chosen at each instant of time $t \geqslant 0$ while simultaneously using the values $z(t)$ and $u(t)$ of the vectors z and u at the same instant of time t.

If, for any value $z(0) \in M$, the game can be conducted in such a way that the point $z(t)$, $t > 0$ never reaches the set M, we shall say that evasion is possible in game (1.1).

We will denote the orthogonal complement of the subspace M in the space R^n by L. Suppose W is, for the present, an arbitrary vector subspace of the space L. We will denote the operation of orthogonal projection from R^n onto W by π. Suppose A and B are two subsets of the space W. We shall write $A \overset{*}{\subset} B$ if a vector $x \in W$ exists such that $x + A \subset B$.

We shall say that the condition of rotatability is satisfied if a two-dimensional subspace W of the space L exists such that a fixed one-dimensional vector space W_1, for which the inclusion

$$\pi e^{tC} Q \subset W_1 \tag{1.2}$$

holds for all sufficiently small positive t, does not exist in W.

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A parallel translation of any of the set P and Q in the space R^n can be compensated by a change in the vector a. Making use of this, we can assume that $0 \in P$, $0 \in Q$ and that, instead of (1.2), the usual inclusion

$$\pi e^{tC} Q \subset W_1$$

holds.

Theorem. Evasion is possible in game (1.1) when the rotatability condition is satisfied and $P \subset Q$.

2. AUXILIARY ASSERTIONS

We define the sets

$$M_{l+1} = \{z \in M_l : Cz + a \in M_l\}, \quad l = 1, 2, \dots, n-1 \tag{2.1}$$

where M_1 is the orthogonal complement of W in R^n. It is obvious that $M \subset M_1$, $M_{l+1} \subset M_l (l = 1, 2, \dots, n-1)$. It can be proved that: 1) if $M_{l+1} = M_l$ for a certain $l = 1, 2, \dots, n-1$, then $M_l = M_{l+l} = \dots = M_n$; 2) if $M_{l+1} \neq M_l$ for all $l = 1, 2, \dots, n-1$, then $n-2 = \dim M_1 > \dim M_2 > \dots > \dim M_{n-1} = 0$ and $M_n = \varnothing$. Hence, it follows that, if $M_n \neq \varnothing$, then $M_{n-1} = M_{n-2}$. This means that a natural number k, $k \leqslant n-2$, exists such that either $M_k \neq M_{k+1} = \varnothing$ or $M_1 \neq M_2 \neq \dots \neq M_k = M_{k+1} = \dots = M_n \neq \varnothing$.

Suppose $\dim M_i = k_i$, A_i is a certain linear, one-to-one mapping of the space R^{k_i} onto M_i, that is

$$A_i R^{k_i} = M_i, \quad N_i = M_i \setminus M_{i+1} \tag{2.2}$$

Henceforth, $i = 1, 2, \dots, k$.

According to Cauchy's formula for the solution of Cauchy's problem

$$\dot{w} = Cw + a, \; w(0) = w_0$$

we have

$$w(\tau, w_0) = e^{\tau C} w_0 + \int_0^{\tau} e^{(\tau - s)C} a ds, \quad -\infty < \tau \leqslant 0 \tag{2.3}$$

Using the set N_i and relation (2.3), we define the surface Σ_i, the equation of which in parametric form is

$$\Sigma_i : w = w(\tau, w_0), \quad \tau \in (-\infty; 0), \quad w_0 \in N_i$$

or

$$\Sigma_i : w = w(\tau, A_i x), \quad \tau \in (-\infty; 0), \quad x \in R^{k_i} \setminus A_i^{-1} M_{i+1} \tag{2.4}$$

Note that the set of points $w = w(\tau, w_0)$, $\tau \in (-\infty; 0)$, $w_0 \in N_k$ can turn out to be a point. In all cases, we call it a surface.

We will denote by $n(z)$ a vector from W such that

$$(n(z), Cz + a) = 0, \quad z \in R^n \tag{2.5}$$

((x, y) is the scalar product of the vectors x and y).

Lemma 1. The surface Σ_i is smooth. If $w(\tau, w_0) \in \Sigma_i$, then the vector

$$m(\tau, w_0) = (e^{-\tau C})^T n(w_0) \tag{2.6}$$

is normal to the surface Σ_i at the point $w(\tau, w_0)$ (T is the sign of transposition).

Proof. Suppose $w_0 = A_i x_0$, $x_0 \in R^{ki} \setminus A_i^{-1} M_{i+1}$.
We calculate (see (2.1)–(2.3) (everywhere henceforth $\partial_\tau = \partial/\partial\tau$, $\partial_j = \partial/\partial x_j$, $j = 1, 2, \dots, k$)

$$\partial_\tau w(\tau, A_i x)|_{x=x_0} = e^{\tau C}(CA_i x_0 + a) = e^{\tau C}(Cw_0 + a)$$
$$\partial_j w(\tau, A_i x)|_{x=x_0} = e^{\tau C} A_i e_j \tag{2.7}$$

Here, $(x_1, x_2, \ldots, x_{k_i})^T = x \in R^{ki}$ and $e_1, e_2, \ldots, e_{k_i}$ are unit coordinate vectors in R^{ki}. It is obvious that the vectors $A_i e_1, A_i e_2, \ldots, A_i e_{k_i}$ form a basis in the space M_i. Hence, taking account of the definition of the sets M_i, N_i, the surface Σ_i and the fact that det $(e^{\tau C}) \neq 0$, we conclude that the vectors

$$\partial_\tau w(\tau, A_i x)|_{x=x_0},\quad \partial_j w(\tau, A_i x)|_{x=x_0}$$

are linearly independent. This means that the surface Σ_i is smooth.

The proof of the second part of the lemma follows from relations (2.5)–(2.7), according to which we have

$$(\partial_\tau w(\tau, A_i x)|_{x=x_0},\ m(\tau, w_0)) = (e^{\tau C}(Cw_0 + a),\ (e^{-\tau C})^T n(w_0)) = 0 \tag{2.8}$$
$$(\partial_j w(\tau, A_i x)|_{x=x_0},\ m(\tau, w_0)) = (e^{\tau C} A_i e_j,\ (e^{-\tau C})^T n(w_0)) = 0$$

Suppose n_0 is an arbitrary, non-zero vector from W. The vector $(e^{-\tau C})^T n_0$ is then normal to the surface $\Sigma_2, \Sigma_3, \ldots, \Sigma_k$ at the point $w(\tau, w_0)$, $w_0 \in N_i$ (see (2.1), (2.4), (2.5) and (2.6)). This means that the vector $(e^{-\tau C})^T n_0$ is normal to the surface $\Sigma_0 = \Sigma_2 \cup \Sigma_3 \cup \ldots \cup \Sigma_k$ at the point $w(\tau, w_0)$, $w_0 \in M_2$. Note that the equation of the surface Σ_0 can be written in the form

$$\Sigma_0 \colon w = w(\tau, w_0),\quad \tau \in (-\infty; 0),\quad w_0 \in M_2$$

For a fixed $(\tau_0, w_0) \in (-\infty; 0)$, $w_0 \in M_2$, we determine the function $v(u)$, $u \in P$ from the equality

$$\max_{v \in Q} |(m(\tau_0, w_0), -u + v)| = |(m(\tau_0, w_0), -u + v(u))|,\quad v(u) \in P \tag{2.9}$$

so that, in the case of an arbitrary measurable function $u(t)$, $a \leqslant t \leqslant b$, $u(t) \in P$, the function $v(u(t))$, $a \leqslant t \leqslant b$ will be measurable,

Lemma 2. Suppose

$$z_0 = w(\tau_0, w_0) \in \Sigma,\ (m(\tau_0, w_0), Q) = \{0\} \tag{2.10}$$

where $\Sigma = \Sigma_1$ or $\Sigma = \Sigma_0$. Then, for any measurable function $u(\cdot)$: $[0; t_0] \to P(t_0 > 0)$, the solution $z(t, z_0, u(\cdot))$ of the equation

$$\dot z = Cz - u + v(u) + a,\quad z(0) = z_0$$

will "descend" from the surface Σ, that is, an instant of time $t_1 \in (0; t_2)$ exists such that $z(t_1, z_0, u(\cdot)) \,\overline{\in}\, \Sigma$.

Proof. We will assume that a measurable function $u_0(\cdot)$: $[0; t_0] \to P$ exists such that

$$z(t, z_0, u_0(\cdot)) \in \Sigma,\quad 0 \leqslant t \leqslant t_0 \tag{2.11}$$

By virtue of the continuity of the vector-function $m(\tau, w)$, $-\infty < \tau < \infty$, $w \in M_1$, the existence of a number $\varepsilon > 0$ such that

$$\min_{u \in P} |(m(\tau, w), -u + v(u))| > 0 \text{ when } |\tau - \tau_0| < \varepsilon,\ |w - w_0| < \varepsilon \tag{2.12}$$

follows from relations (2.9)–(2.11).

Next, it follows from inclusion (2.10) that

$$z(t, z_0, u_0(\cdot)) \in \{w(\tau, w_1)\colon\ |\tau - \tau_0| < \varepsilon,\ w_1 \in w_0(\varepsilon)\},\ 0 \leqslant t \leqslant t_2$$
$$w_0(\varepsilon) = \{w \in M_1 \setminus M_2\colon\ |w - w_0| < \varepsilon\} \text{ when } \Sigma = \Sigma_1 \tag{2.13}$$
$$w_0(\varepsilon) = \{w \in M_2\colon\ |w - w_0| < \varepsilon\} \text{ when } \Sigma = \Sigma_0$$

where t_2 is a fairly small positive number.

We will denote the normal to the surface Σ at the point $z = z(t, z_0, u_0(\cdot)) \in \Sigma$ by $h = h(z(t, z_0, u_0(\cdot)))$. Then (see (2.13))

$$h \in \{m(\tau, w) : |\tau - \tau_0| < \varepsilon, \quad w \in w_0(\varepsilon)\}, \quad 0 \leqslant t \leqslant t_2 \tag{2.14}$$

The equality

$$(Cz + a, h) = 0, \quad 0 \leqslant t \leqslant t_2$$

follows from relations (2.8) and (2.11). Hence, we have that

$$(\dot{z}, h) = (Cz - u_0(t) + v(u_0(t)) + a, h) =$$
$$= (-u_0(t) + v(u_0(t)), h) = 0$$

almost everywhere in $[0; t_2]$. The last equality contradicts inequality (2.12) (see the inclusion (2.14)).

3. PROOF OF THE THEOREM

Suppose that, at the instant of time $t = 0$, an object z is located at the point z_0, that is, $z(0) = z_0$. We put $\Sigma = \Sigma_0 \cup \Sigma_1$ (see Lemma 2). There are two possible cases.

A. If the solution $z(t, z_0)$, $t \geqslant 0$ of the equation

$$\dot{z} = Cz + a, \quad z(0) = z_0 \tag{3.1}$$

never reaches the set M, we put $v = v(u) = u$. It is then obvious that the object z never reaches the set M.

B. Suppose the solution $z(t, z_0)$, $t \geqslant 0$ of Eq. (3.1) reaches the set M at a certain instant of time $t = t_0$, $t_0 > 0$ (it can be assumed that $z(t, z_0) \overline{\in} M$, $0 \leqslant t \leqslant t_0$ without any loss of generality). We put $z(t_0, z_0) = w_0$. It is then obvious that $w_0 \in M$ and that the trajectories $z(t, z_0)$, $0 \leqslant t \leqslant t_0$ and $w(\tau, w_0)$, $-t \leqslant \tau \leqslant 0$ are identical: $z(t, z_0) = w(t - t_0, w_0)$, where $w(\tau, w_0)$ is the solution of the equation

$$\dot{w}(\tau) = Cw(\tau) + a, \quad w(0) = w_0$$

We assume that

$$(m(\tau, w_0), Q) \equiv \{0\}, \quad -t_0 \leqslant \tau < 0$$

Then (see (2.6))

$$\{0\} = (m(\tau, w_0), Q) = ((e^{-\tau C})^T n(w_0), Q) = (n(w_0), \ e^{-\tau C} Q) = (n(w_0), \pi e^{-\tau C} Q), \quad -t_0 \leqslant \tau < 0$$

Hence, on taking account of the inclusion $n(w_0) \in W$, we conclude that the sets lie in the one-dimensional subspace $W_1 \in W$ for all $\tau \in [-t_0, 0]$, which contradicts the rotatability condition. This means that a number τ_0, $\tau_0 \in [-t_0, 0]$ exists such that $(m(\tau_0, w_0), Q) \neq \{0\}$. We put $v(t) = u(t)$ in the time interval $[0; t_0 + \tau_0]$. Then

$$z(t) = w(t - t_0, w_0) \overline{\in} M, \quad 0 \leqslant t \leqslant t_0 + \tau_0$$

Hence, without any loss of generality, we assume that $\tau_0 = -t_0$, that is

$$(m(-t_0, w_0), Q) = (m(\tau_0, w_0), Q) \neq \{0\}$$

We now put $z_0(\sigma) = \{z \in R^n \colon |z - z_0| < \sigma\}$ and choose the number σ so that

1) $z_0(\sigma) \cap M = \varnothing$, 2) $z_0(\sigma) \cap \Sigma_0 = \varnothing$ if $z_0 \in \Sigma_1$.

It can be shown that it is possible to choose such a σ. Next, we determine the function $v = v_0(u)$, $u \in P$ from equality (2.9).

We fix the positive number ε such that, for an arbitrary measurable function $u(t)$, $0 \leqslant t \leqslant \varepsilon$, $u(t) \in P$, the inclusion $z \in z_0(\sigma)$, $0 \leqslant t \leqslant \varepsilon$ is satisfied, where $z = z(t, z_0, u(\cdot), v_0(u(\cdot)))$ is the solution of the equation

$$\dot{z} = Cz - u(t) + {}_0(u(t)) + a, \quad z(0) = z_0$$

It is obvious that the number ε_0 exists.

Suppose the pursuer chooses an arbitrary control $u = u(t)$, $t \geq 0$. We assume that

$$v(t) = v_0(u(t)), \quad t \geq 0$$

Then, by virtue of Lemma 2, a number t_1, $0 < t_1 < \varepsilon$ exists such that

$$z(t_1, z_0, u(\cdot),\ v_0(u(\cdot))) \overline{\in} \Sigma_m$$

if $z_0 \in \Sigma_m$, $m = 0, 1$. Moreover, if $z_0 \in \Sigma_1$, then, by virtue of the choice of the numbers σ and ε, we have

$$z(t_1, z_0, u(\cdot),\ v_0(u(\cdot))) \overline{\in} \Sigma_0 \cup \Sigma_1 = \Sigma$$

This means that, if $z_0 \in \Sigma_0$, then, in a sufficiently small time interval, we can get that $z(t_1) \overline{\in} \Sigma_0$. If, at the same time, it is found that $z(t_1) \in \Sigma_1$, then, in a similar manner, we get that

$$z(t_2) \overline{\in} \Sigma_0 \cup \Sigma_1 = \Sigma$$

Here, $z(t) \overline{\in} M$ for all $t \in [0; t_2]$.

When $t \geq t_2$, we put $v = v(u) = u$ and the point z, by moving in accordance with relation (2.3), does not fall in the set M.

The theorem is proved.

4. EXAMPLE

In game (1.1), $n \geq 2$ and the sets P, Q and M have the form

$$P = \{x = (x_1, x_2, \ldots, x_n)^T \in R^n :\ |x_i| \leq 1/\sqrt{n},\ i = 1, 2, \ldots, n\}$$

$$Q = \{x = (x_1, x_2, \ldots, x_n)^T \in R^n :\ x_1^2 + x_2^2 + \ldots + x_n^2 \leq 1\}$$

$$M = \{x = (x_1, x_2, \ldots, x_n)^T \in R^n :\ x_3 = x_4 = \ldots = x_n = 0\}$$

($M = \{0\}$ when $n = 2$). It is then obvious that the set $e^{\tau C}Q$ is a body in R^n and that $\pi e^{\tau C}Q$ contains an interior point as a two-dimensional body, that is, the condition of rotatability is satisfied. Moreover, $P \subset Q$ and $\mu = 1$. At the same time, Pontryagin's conditions and those in [2–10] are not satisfied.

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